THE GAMMA AND BETA FUNCTIONS

Notes and Problems
Designed for use in Mathematical Statistics
and Mathematical Physics

By

W. EDWARDS DEMING

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Definition of the Gamma function.-- The Gamma function for positive values of n can be defined by the integral[1]

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, dx. \tag{1}$$

$\Gamma(n)$ is read "the Gamma function of n." It is truly a function of n, that is, it depends on n for its value. With the upper limit infinite, as here written, the Gamma functions is said to be complete, but this adjective is usually omitted except when it is desired to distinguish a complete from an incomplete Gamma function. The incomplete Gamma function is defined by[2]

$$\Gamma_x(n) = \int_0^x x^{n-1} e^{-x} \, dx, \tag{2}$$

which is the same integral except that the upper limit is a variable. The incomplete Gamma function $\Gamma_x(n)$ is thus

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[1] The integral given here as the definition of the Gamma function was studied by Euler, mainly in the form $\int_0^1 (\ln 1/x)^\lambda \, dx$ (vol. 4 of his Institutionis Calculi Integralis; Petrograd, 1770), and is sometimes referred to as "Euler's second integral." An excellent presentation of the Euler integrals is given by Joseph Edwards in his Integral Calculus, vol. 2, Ch. 24 (Macmillan, 1922). In the theory of the functions of a complex variable, other definitions of the Gamma function are also given, all of which are equivalent to the Euler integral in Eq. (1) when the real part of n is positive.

[2] An integral with limits is a function of those limits, and not a function of the variable used in the integrand. In Eq. (2) the subscript x in $\Gamma_x(n)$ is to be identified with the upper limit of the integral, not with the x in the integrand. The student may find it helpful to rewrite Eq. (2) as $\Gamma_x(n) = \int_0^x t^{n-1} e^{-t} \, dt$. Any letter whatever may be used in the integrand without affecting the value of the integral, provided its limits remain 0 and x. Similar remarks apply to Eq. (14).

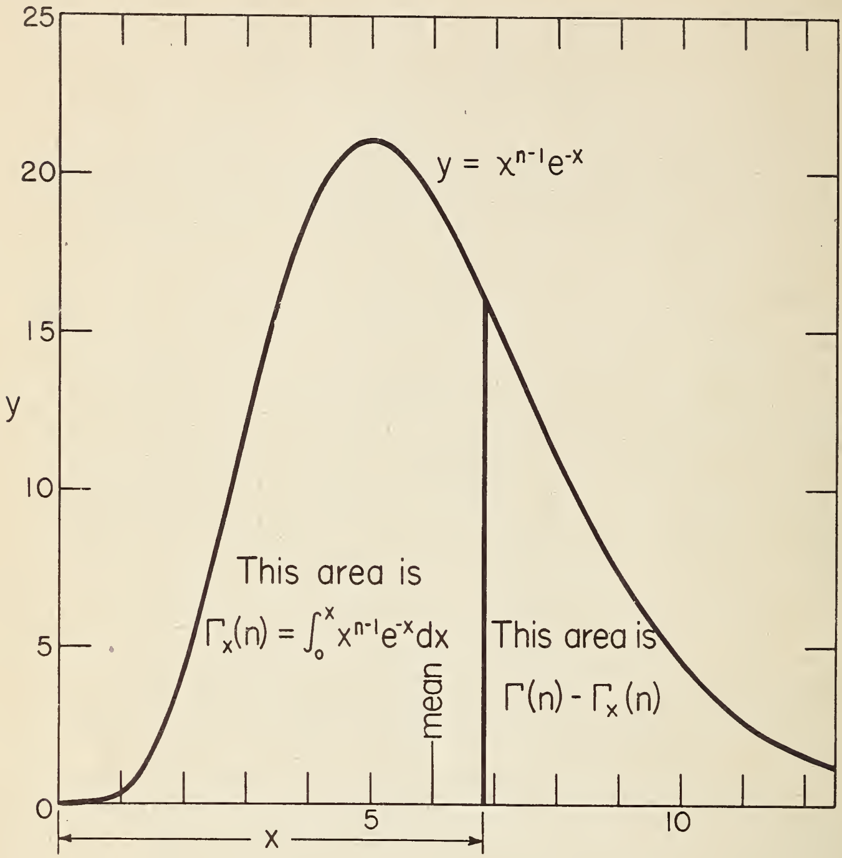

Fig. 1. The curve $y = x^{n-1}\ e^{-x}$, this being the integrand in the definition of the Gamma function. The curve is drawn for $n = 6$. The area under the whole curve to infinity is the complete Gamma function $\Gamma(n)$, and the area to the finite abscissa x is the incomplete Gamma function $\Gamma_x(n)$.

not only a function of n but also of x. For consistency the complete function $\Gamma(n)$ should have the subscript ∞, but for brevity and by convention the subscript is generally omitted.

It is instructive to plot the integrand of Eqs. (1) and (2) for a given value of n, for positive values of x. The area under the whole curve to infinity is by definition the complete function $\Gamma(n)$. The area included between 0

and the finite abscissa x is $\Gamma_x(n)$. The exact shape of the curve varies with the parameter n, but the relation between the complete and incomplete functions is always as shown in Fig. 1. The reader may agree that the adjective incomplete is aptly applied here; the Gamma function is incomplete when the integration extends only part way, i.e., to some finite distance.[3]

Other forms of the above integrals.--The above integrals can take a variety of other forms[4] by change of variable. In the theory of errors an important form is obtained by setting $x = u^2$; then $dx = 2\,u\,du$, and

$$\Gamma_x(n) = \Gamma_{u^2}(n) = 2\int_0^{u=\sqrt{x}} u^{2n-1} e^{-u^2} du. \tag{3}$$

The student should take careful note that in the incomplete Gamma function the subscript of Γ always refers back to the upper limit[2] of the integral in Eq. (2); thus the subscript x or u^2 in (3) is the upper limit of the integral in Eq. (2) and not of the new form (3).

With $n = \frac{1}{2}$, Eq. (3) gives

$$\Gamma_x(\tfrac{1}{2}) = 2\int_0^{\sqrt{x}} e^{-x^2} dx, \tag{4}$$

whence we perceive that the normal probability integral can be regarded as a table of the incomplete function $\Gamma_x(\frac{1}{2})$.

Several other forms of the integral for the Gamma function that are important in application will be found in the examples at the end of this chapter.

The recursion formula.--The most important property of the Gamma function is a recursion formula. It can be found by integrating the right-hand side of Eq. (1) by parts, thus:

[3] Several ways of evaluating the incomplete Gamma function by series and continued fractions will be found in Exercises 27 and 28.

[4] See, for example, E. B. Wilson, Advanced Calculus, Ch. 14 (Ginn and Company, 1912).

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx = (1/n) \int_0^\infty e^{-x}\, dx^n \qquad (n > 0)$$

$$= (1/n) \{[x^n e^{-x}]_0^\infty + \int_0^\infty x^n e^{-x}\, dx\}$$

$$= (1/n) \{0 + \Gamma(n+1)\}.$$

Accordingly, there follows the <u>recursion formula</u>

$$\Gamma(n+1) = n\,\Gamma(n). \tag{5}$$

Thus $\Gamma(n+1)$ is simply related to $\Gamma(n)$. Eq. (5) may be regarded as a difference equation, and the Gamma function as a solution thereof.

By repeated applications of the recursion formula it is seen that

$$\Gamma(n+1) = n\,\Gamma(n) = n\,\Gamma(n-1+1) = n(n-1)\,\Gamma(n-1)$$

$$= n\,(n-1)\,(n-2)\,\ldots\,(n-k)\,\Gamma(n-k). \tag{6}$$

Hence if the Gamma function is tabled over a unit interval of the argument, the function for arguments outside this range can be found; for example

$$\Gamma(3.37) = \Gamma(2.37+1) = 2.37 \times \Gamma(2.37) = 2.37 \times \Gamma(1.37+1)$$

$$= 2.37 \times 1.37\,\Gamma(1.37), \tag{7}$$

so that if $\Gamma(1.37)$ is known, then $\Gamma(3.37)$ can be evaluated by a simple multiplication.

The recursion formula was here derived from the definition contained in Eq. (1), and it was necessary to postulate that $n > 0$; otherwise the term $[x^n e^{-x}]_0^\infty$ would not have dropped out in the integration by parts. But the recursion formula, it so happens, is a general property of the Gamma function however defined, and it holds for negative and complex values of the argument as well as for positive real values.

<u>The relation of the Gamma function to factorials</u>.--With $n = 1$, Eq. (1) gives

$$\Gamma(1) = \int_0^\infty e^{-x}\, dx = 1.$$

Hence if n is a positive integer, and if Eq. (6) is extended, the recursion formula leads to the following simple and important relation with the factorial:

$$\Gamma(n+1) = n(n-1)(n-2) \ldots 5\cdot4\cdot3\cdot2\cdot1\ \Gamma(1) = n! \qquad (8)$$

For values of n not integral, $\Gamma(n+1)$ can be regarded as a generalized n!; the practice is in fact fairly common. Thus $\frac{3}{4}!$ is to be identified with $\Gamma(\frac{7}{4})$; $-\frac{1}{2}!$ with $\Gamma(\frac{1}{2})$; etc.

Tables of the complete Gamma function.--Gauss[5] in 1813 computed log n! to twenty places and d ln n!/dn to eighteen places in the interval $0 \leqq n \leqq 1$ by steps of 0.01. Degen[6] in 1824 computed log n! to eighteen decimals from n = 1 to n = 1200 by integral steps. Legendre[7] in 1825 gave log $\Gamma(n)$ to twelve places for $1 \leqq n \leqq 2$ by steps of 0.001, together with its first, second, and third differences. Table XXXI in Tables for Statisticians and Biometricians,[8] Part I, shows log $\Gamma(n)$ to seven places in the interval $1 \leqq n \leqq 2$ at intervals of 0.001, and Table XLIX in the same volume shows log n! to seven decimals for every integer from 1 to 1000 inclusive. In 1922 Egon Pearson[9] published a table of log $\Gamma(n)$ to ten decimals from 2 to 1200 inclusive, by steps varying from a tenth to a whole integer. In 1923 John Brownlee[10] published a table of log $\Gamma(n)$ to seven places by steps of 0.01 in the interval $1 \leqq n \leqq 50.9$. It is now not unusual to find abbreviated tables of n! or its logarithm in tables for aids in computation.

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[5]Carl Friederich Gauss, Werke, vol. 3, pp. 161-162. Gauss' notation is πz for "Gauss' infinite product," equivalent to z! He introduced ψ_z for the derivative $\frac{d}{dz} \ln \pi z$.

[6]Carl Ferdinand Degen, Tabularem ad faciliorem et breviorem Probabilitatis computationem utilium Enneas (Copenhagen, 1824).

[7]Legendre, Traité de fonctions elliptiques, vol. 2, pp. 490-499, (Paris, 1825). A facsimile of Legendre's table has been issued as Tracts for Computers, No. 4 (Cambridge, 1921).

[8]Tables for Statisticians and Biometricians comes in two volumes. Part I first appeared in 1914, with second and third editions in 1924 and 1930. Part II appeared in 1931. Both volumes are issued from The Office of Biometrika, University College, Gower Street, London W.C. 1.

[9]Egon S. Pearson, Tracts for Computers, No. 8 (Cambridge, 1922).

[10]John Brownlee, Tracts for Computers, No. 9 (Cambridge, 1923).

The graph of the Gamma function.--It is easy to evaluate the Gamma function at any positive integer with Eq. (8). For example, $\Gamma(4) = 3! = 6$ and $\Gamma(5) = 4! = 24$. The curve is continuous on the positive side of the vertical axis, as can be proved,[11] hence a graph like Fig. 2 is not difficult to construct for positive values of n. The minimum between n = 1 and n = 2 is interesting, and requires special analysis. We have already noticed that

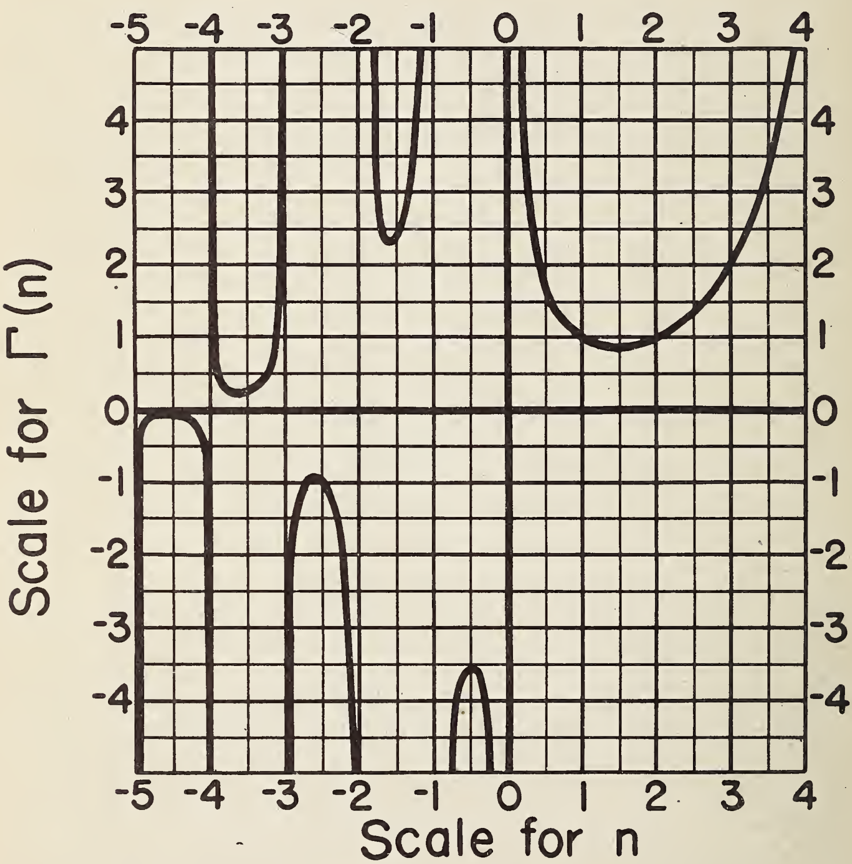

Fig. 2. Graph of the Gamma Function.

[11] See, for example, H. S. Carslaw, Fourier's Series and Integrals, pp. 132 and 133 in the 1930 edition (Macmillan, 1906, 1921, 1930).

$\Gamma(1) = 1$, and by Eq. (8) it follows that $\Gamma(2) = 1! = 1$ also; hence there must be a proper minimum somewhere between. Calculations have located the position of the minimum at $n = 1.46163\ 21449\ 68\ \ldots$, at which point the Gamma function has the value $0.88560\ 31944\ \ldots$.

For negative values of n the integral in Eq. (1) is not used to define the Gamma function; instead, other definitions are used, such as Gauss' (below) and Weierstrass'. The recursion formula permits evaluation of the Gamma function for negative n in terms of its values for a positive n. Exercises 5 and 6 at the end give hints on ways of discovering the shape of the graph of the Gamma function where n is negative. As might be inferred from Fig. 2, when n is a negative integer the Gamma function suffers infinite discontinuities in the nature of simple poles, becoming alternately positive and negative infinity. Accordingly, the factorial of a negative integer such as -1!, -2!, -3!, etc. is infinite, and its reciprocal is zero. This phenomenon appears in Exercises 13 and 14.

A very interesting chapter on the subject is found in Edwards' Integral Calculus (Macmillan, 1922), Ch. XXIV in volume 2, sections 886 and 922 in particular. Edwards shows graphs of

$$y = 1/(n+1)$$
$$y = 2!2^n/(n+1)(n+2)$$
$$y = 3!3^n/(n+1)(n+2)(n+3)$$
$$y = 4!4^n/(n+1)(n+2)(n+3)(n+4).$$

By continuation, this sequence of curves can be made to approach $\Gamma(n+1)$ for any n as closely as desired, wherefore the successive curves give better and better approximations to the graph in Fig. 2 for finite values of n. This sequence, continued indefinitely, is in fact the Gauss definition of $\Gamma(n+1)$--Gauss' infinite product,[5] identified as n!

Tables of the incomplete Gamma function.--By transformation of variable, as called for in Exercises 32-42, several important curves (the Pearson Types III and V, and others) can be changed into the form $y = x^{n-1} e^{-x}$, the integrand of Eq. (1), wherefore the incomplete Gamma function constitutes an important probability integral. The need of tables of this integral was recognized by Karl Pearson, who with the assistance of his staff at the Biometric Laboratory, turned out in 1921, after many years' labor, the

Tables of the Incomplete Gamma Function.[12] A sample page is shown in Fig. 3. Therein the function $I(u,p)$ is tabled to seven decimals against the arguments u and p, along with second and fourth differences with respect to both u and p. The function $I(u,p)$ is related to u and p, and through them to the incomplete Gamma function, by the definition

$$I(u,p) = \Gamma_x(p+1)/\Gamma(p+1) = \int_0^x x^p e^{-x}\,dx \Big/ \int_0^\infty x^p e^{-x}\,dx \quad (9)$$

wherein $u = x/\sqrt{p+1}$.

Evidently for any fixed value of p, $I(u,p)$ varies from 0 to 1 as x varies from 0 to ∞; $I(u,p)$ is in fact just the fractional part of the area under the curve of Fig. 1 between 0 and the abscissa x, p being identified as $n-1$, and u as $x/\sqrt{p+1} = x/\sqrt{n}$. The advantage of tabling against $x/\sqrt{n}$ instead of against x directly arises from the fact that as n increases the curve of Fig. 1 flattens out, and one is accordingly compelled to move to higher and higher values of x if he would include always a specified fraction of the area. The device of using $x/\sqrt{n}$ amounts to a change of scale that keeps the table within bounds; when n is high, a moderate value of $x/\sqrt{n}$ corresponds to a high value of x. $x/\sqrt{n}$ is in fact just the upper limit x of the incomplete Gamma function (Eq. 2) expressed in units of the standard deviation $\sqrt{n}$ of the curve. (Exercises 1 and 29 will illuminate this point. It may be noted here that the mean of the curve of Fig. 1 lies at abscissa n, and the standard deviation of the curve is $\sqrt{n}$.) The introductory material in the Tables of the Incomplete Gamma Function gives a history of the work and instructions for the use of the tables, together with illustrative examples.

In 1930 L. R. Salvosa[13] published an extensive table of the areas and ordinates of the Pearson Type III curve

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[12] Tables of the Incomplete Gamma Function, published in 1922 by his Majesty's Stationery Office, and reissued in 1934 by the Office of Biometrika, University College, London W.C. 1.

[13] L. R. Salvosa, Ann. Math. Statistics, vol. 1, 1930; pp. 191-225.

18 $u = 0{\cdot}0$ to $6{\cdot}2$ TABLES OF THE INCOMPLETE Γ-FUNCTION $p = 3{\cdot}0$ to $3{\cdot}5$

	$p = 3{\cdot}0$			$p = 3{\cdot}1$			$p = 3{\cdot}2$			$p = 3{\cdot}3$			$p = 3{\cdot}4$			$p = 3{\cdot}5$	
u	$I(u, p)$	δ_u^2 δ_u^4	δ_p^2 δ_p^4	$I(u, p)$	δ_u^2 δ_u^4	δ_p^2 δ_p^4	$I(u, p)$	δ_u^2 δ_u^4	δ_p^2 δ_p^4	$I(u, p)$	δ_u^2 δ_u^4	δ_p^2 δ_p^4	$I(u, p)$	δ_u^2 δ_u^4	δ_p^2 δ_p^4	$I(u, p)$	u
·0	·0000000	—	0	·0000000	—	0	·0000000	—	0	·0000000	—	0	·0000000	—	0	·0000000	·0
·1	·0000568	+6626	+40 +1	·0000436	+5494	+30	·0000334	+4549	+24	·0000256	+3759	+18	·0000196	+3102	+14	·0000150	·1
·2	·0007762	+18625 +775	+298 +10	·0006366	+16304 +1417	+247 +9	·0005217	+14237 +1934	+204 +8	·0004271	+12407 +2320	+169 +6	·0003494	+10788 +2608	+139 +5	·0002855	·2
·3	·0033581	+31399 −2308	+832 +16	·0028600	+28531 −1648	+713 +14	·0024337	+25859 −1050	+618 +13	·0020693	+23375 −494	+533 +12	·0017580	+21082 −9	+457 +11	·0014924	·3
·4	·0090799	+41865 −3606	+1563 +21	·0079365	+39110 −3148	+1384 +19	·0069316	+36431 −2682	+1225 +17	·0060490	+33849 −2238	+1082 +16	·0052748	+31367 −1798	+956 +14	·0045961	·4
·5	·0189882	+48725 −3830	+2331 +19	·0169240	+46541 −3594	+2128 +18	·0150726	+44321 −3332	+1924 +17	·0134136	+42085 −3043	+1737 +16	·0119283	+39854 −2741	+1565 +15	·0105995	·5
·6	·0337690	+51755 −3466	+3080 +16	·0305656	+50378 −3404	+2835 +15	·0276457	+48879 −3310	+2609 +14	·0249867	+47278 −3183	+2395 +13	·0225672	+45600 −3038	+2200 +12	·0203677	·6
·7	·0537253	+51319 −2816	+3660 +13	·0492450	+50811 −2880	+3420 +12	·0451067	+50127 −2909	+3192 +11	·0412876	+49288 −2912	+2976 +10	·0377661	+48308 −2875	+2770 +10	·0345216	·7
·8	·0788135	+48067 −2083	+4046 +8	·0730055	+48364 −2215	+3829 +9	·0675804	+48466 −2321	+3620 +9	·0625173	+48386 −2401	+3416 +10	·0577958	+48141 −2467	+3223 +10	·0533966	·8
·9	·1087084	+42732 −1367	+4224 +3	·1016024	+43702 −1531	+4044 +3	·0949007	+44484 −1683	+3866 +3	·0885856	+45083 −1813	+3691 +4	·0826396	+45507 −1920	+3519 +4	·0770454	·9
1·0	·1428765	+36030 −744	+4207 −1	·1345695	+37509 −914	+4069 +0	·1266694	+38819 −1066	+3930 0	·1191622	+39967 −1216	+3790 0	·1120341	+40953 −1354	+3651 +1	·1052710	1·0
1·1	·1806476	+28584 −220	+4022 −4	·1712875	+30402 −375	+3927 −3	·1623200	+32088 −536	+3829 −3	·1537355	+33635 −681	+3729 −2	·1455239	+35045 −821	+3626 −2	·1376749	1·1
1·2	·2212771	+20918 +177	+3705 −4	·2110457	+22920 +44	+3654 −3	·2011794	+24821 −81	+3592 −3	·1916723	+26622 −218	+3530 −3	·1825182	+28316 −352	+3464 −3	·1737105	1·2
1·3	·2639984	+13429 +474	+3292 −4	·2530959	+15482 +367	+3276 −4	·2425209	+17478 +250	+3253 −3	·2322713	+19391 +142	+3224 −2	·2223441	+21235 +31	+3192 −1	·2127361	1·3
1·4	·3080626	+6414 +666	+2823 −5	·2966943	+8411 +588	+2837 −5	·2856097	+10375 +508	+2843 −5	·2748094	+12302 +418	+2844 −5	·2642935	+14185 +323	+2842 −4	·2540618	1·4
1·5	·3527682	+65 +789	+2329 −7	·3411338	+1928 +734	+2366 −7	·3297360	+3785 +673	+2395 −6	·3185777	+5631 +606	+2420 −6	·3076614	+7458 +538	+2440 −6	·2969891	1·5
1·6	·3974803	−5495 +839	+1839 −7	·3857661	−3821 +803	+1889 −7	·3742408	−2132 +766	+1936 −6	·3629091	−434 +726	+1977 −6	·3517751	+1269 +678	+2015 −6	·3408426	1·6
1·7	·4416429	−10216 +840	+1369 −5	·4300163	−8767 +832	+1427 −5	·4185324	−7283 +809	+1486 −5	·4071971	−5773 +781	+1539 −6	·3960157	−4242 +751	+1589 −6	·3849932	1·7
1·8	·4847839	−14097 +812	+934 −4	·4733898	−12881 +807	+1000 −4	·4620957	−11625 +805	+1062 −4	·4509078	−10331 +799	+1122 −5	·4398321	−9002 +783	+1178 −5	·4288742	1·8
1·9	·5265152	−17166 +755	+545 −3	·5154752	−16188 +766	+613 −3	·5044965	−15162 +771	+676 −4	·4935854	−14090 +770	+740 −4	·4827483	−12979 +773	+798 −4	·4719910	1·9
2·0	·5665299	−19480 +683	+209	·5559418	−18729 +697	+274	·5453811	−17928 +711	+336	·5348540	−17079 +724	+397	·5243666	−16183 +730	+457	·5139249	2·0
2·1	·6045966	−21111 +601	−73	·5945355	−20573 +627	−15	·5844729	−19983 +640	+44	·5744147	−19344 +662	+99	·5643666	−18657 +676	+159	·5543344	2·1
2·2	·6405522	−22141 +520	−305	·6310719	−21790 +540	−252	·6215664	−21392 +564	−199	·6120410	−20947 +587	−146	·6025009	−20455 +607	−95	·5929514	2·2
2·3	·6742937	−22651 +437	−488	·6654293	−22467 +465	−442	·6565207	−22237 +486	−395	·6475726	−21963 +507	−348	·6385897	−21646 +530	−301	·6295767	2·3
2·4	·7057701	−22724 +361	−627	·6975400	−22679 +386	−586	·6892513	−22596 +409	−547	·6809079	−22472 +432	−506	·6725139	−22309 +457	−466	·6640733	2·4

Fig. 3. A sample page in the Tables of the Incomplete Gamma Function.

$$\left.\begin{aligned} y &= y_0(1+\tfrac{1}{2}\alpha t)^{\frac{4}{\alpha^2}-1}\, e^{-2t/\alpha} \\ \text{wherein}\quad y_0 &= (\tfrac{4}{\alpha^2})^{\frac{4}{\alpha^2}-\frac{1}{2}} \Big/ e^{4/\alpha^2}\, \Gamma(4/\alpha^2) \end{aligned}\right\} \tag{10}$$

the arguments being α and t. $\frac{1}{2}\alpha$ is the "skewness" of the curve, and t the abscissa measured from the mean. Salvosa's table can be regarded as another form of a table for the incomplete Gamma function. (See Exercise 46.)

Tables of "chi-square" can also be used to evaluate the incomplete Gamma function. Some of these tables show

$$P(\chi) = \frac{1}{\Gamma(\frac{1}{2}k)2^{\frac{1}{2}k}} \int_{\chi^2}^{\infty} (\chi^2)^{\frac{1}{2}k-1} e^{-\frac{1}{2}\chi^2}\, d\chi^2 \tag{11}$$

tabulated against χ^2 and k as arguments. Other tables, following Fisher, show χ^2 tabulated against $P(\chi)$ and k as arguments. It can readily be shown by a change of variable that

$$P(\chi) = 1 - \Gamma_{\frac{1}{2}\chi^2}(\tfrac{1}{2}k)/\Gamma(\tfrac{1}{2}k). \tag{12}$$

(See Exercises 34 and 42.)

Another notation for the incomplete Gamma function, used occasionally by continental writers, is $(n,x)!$ for $\Gamma_x(n+1)$. In this symbolism $(n,\infty)!$ means $n!$

Besides supplying tables for integrals under the Type III curve and other curves reducible thereto, the incomplete Gamma function affords a method of summing any number of terms of the Poisson exponential limit without approximation, as is demonstrated in a later section (p. 20).

Definition of the Beta function.--The Beta function will here be defined by the integral[14]

[14] The integral given here as the definition of the Beta function was studied by Euler mainly in the form $\int_0^1 x^{m-1}(1-x^n)^{1-k/n}\,dx$, n fixed (Art. 345 in vol. 1 of his Institutiones Calculi Differentialis, Petrograd, 1768), and is sometimes referred to as "Euler's first integral. The relation between the Gamma and Beta functions, shown here as Eq. (24), was given by him in Art. 27 of vol. 4.

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx. \tag{13}$$

$B(m,n)$ is read "the Beta function of m and n." It is truly a function of m and n; that is, it depends on both m and n for its value. With the upper limit unity, as here written, the Beta function is said to be complete, but, as with the Gamma function, the adjective is usually omitted except when there is a possibility of confusing the complete with the incomplete Beta function. The incomplete Beta function is defined by[2]

$$B_x(m,n) = \int_0^x x^{m-1}(1-x)^{n-1}\,dx, \tag{14}$$

the upper limit x lying between 0 and 1. Evidently the incomplete Beta function $B_x(m,n)$ is a function not only of m and n, but also of the upper limit x. In the same symbolism the complete Beta function $B(m,n)$ should have the subscript 1, but for brevity and by convention it is generally omitted, just as the subscript ∞ is omitted in writing $\Gamma(n)$ for the complete Gamma function.

The student should note that the Beta function, complete or incomplete respectively, has one more argument than the Gamma function, complete or incomplete.

Interchanging the arguments m and n.--By change of variable the above integrals assume a variety of forms[4] and lead to several interesting relations, of which perhaps the most important is that

$$B(m,n) = B(n,m). \tag{15}$$

This is easily proved by replacing x in Eq. (13) by $1-y$. If $x = 1-y$, then $dx = -dy$, and

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = -\int_1^0 (1-y)^{m-1}y^{n-1}\,dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy = B(n,m).$$

By the same change of variable

$$B_x(m,n) = \int_{1-x}^1 y^{n-1}(1-y)^{m-1}\,dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy - \int_0^{1-x} y^{n-1}(1-y)^{m-1}\,dy.$$

That is,

$$B_x(m,n) = B(m,n) - B_{1-x}(n,m). \tag{16}$$

Or, in another form,

$$B_x(n,m) = B(m,n) - B_{1-x}(m,n). \tag{17}$$

Hence in the incomplete Beta function $B_x(m,n)$ an interchange of the two arguments alters the value to $B(m,n) - B_{1-x}(m,n)$. That an interchange of the arguments m and n has no effect on the complete Beta function $B(m,n)$ follows at once by putting $x = 1$ in Eqs. (16) or (17) and noting that $B_0(m,n) = B_0(n,m) = 0$.

These relations are easily recalled by using a plot of the integrand of Eqs. (13) and (14), such as Fig. 3. The effect of interchanging m and n is to produce a mirror image of the figure, which can be visualized as the curve that would be obtained by moving the origin one unit to the right, reversing the positive direction of x, and then looking at the page through the back side of the paper. By visualizing this process, the relations exhibited by Eqs. (16) and (17) become evident.

A recursion formula for the incomplete Beta function is given in Exercise 16.

Unless m and n are equal, the curve $y = x^{m-1}(1-x)^{n-1}$ will be unsymmetrical. By varying the choice of m and n, a great variety of forms can be obtained. The student will profit by carrying out the calculations and plotting the curves suggested in Exercise 2.

Two other useful forms of the Beta function.--Let $x = z^2/(1+z^2)$, $1 - x = 1/(1+z^2)$, $dx = 2z\,dz/(1+z^2)^2$. By introducing this change of variable into Eq. (13) we find that

$$B(m,n) = 2\int_0^\infty z^{2m-1}(1+z^2)^{-m-n}dz = B(n,m). \tag{18}$$

In particular, if m is replaced by $\frac{1}{2}$ and n by $\frac{1}{2}[n - 1]$, this relation gives

$$B(\tfrac{1}{2}[n-1], \tfrac{1}{2}) = 2\int_0^\infty (1+z^2)^{-\frac{1}{2}n}dz = \int_{-\infty}^\infty (1+z^2)^{-\frac{1}{2}n}dz, \tag{19}$$

which will be found handy for certain kinds of calculations.

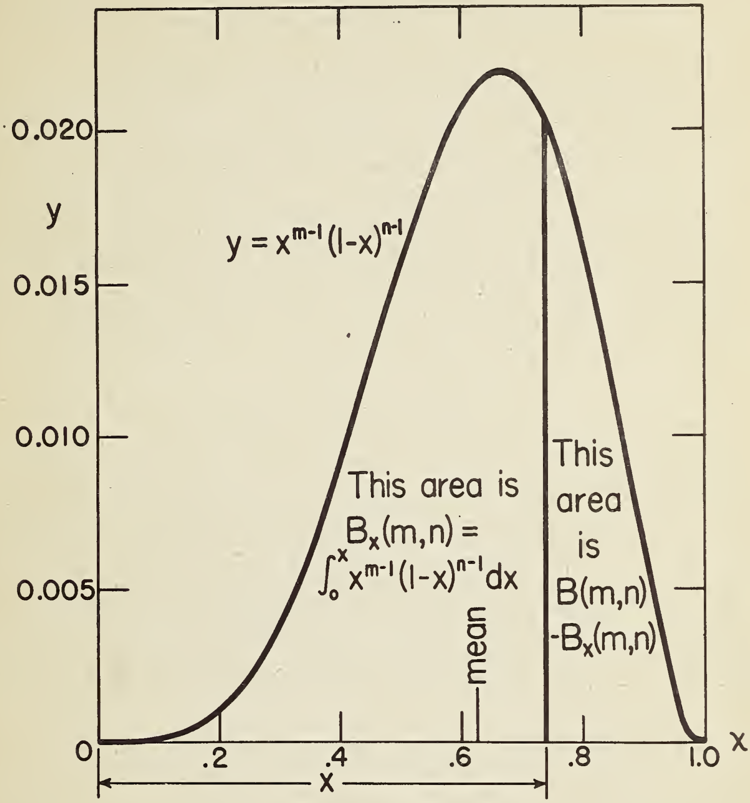

Fig. 4. The curve $y = x^{m-1}(1-x)^{n-1}$, this being the integrand in the definition of the Beta function. The curve is drawn for m = 5, n = 3. The area under the whole curve from 0 to 1 is the complete Beta function B(m,n), and the area between 0 and x < 1 is the incomplete Beta function B(m,n).

Another useful form is obtained by the substitution $x = \sin^2\varphi$, $dx = 2\sin\varphi\cos\varphi\, d\varphi$, whereupon the original form shown in Eq. (13) goes into

$$B'(m,n) = 2\int_0^{\frac{1}{2}\pi} \sin^{2m-1}\varphi \cos^{2n-1}\varphi \, d\varphi. \tag{20}$$

In particular,

$$B(\tfrac{1}{2}, \tfrac{1}{2}) = 2\int_0^{\frac{1}{2}\pi} d\varphi = \pi, \tag{21}$$

a relation that will be needed shortly (see Eq. 26).

The relation between the Gamma and Beta functions.--From the second form of the Gamma function, Eq. (3),

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty u^{2m-1}e^{-u^2}du \cdot \int_0^\infty v^{2n-1}e^{-v^2}dv$$

$$= 4\int_0^\infty\int_0^\infty u^{2m-1}v^{2n-1}e^{-(u^2+v^2)}du\, dv. \qquad (22)$$

It is permissible to change the product of two integrals to a double integral, and vice versa, when neither the limits nor the integrands of either integral depend on the variable in the other.[15]

Now consider the surface

$$z = u^{2m-1}v^{2n-1}e^{-(u^2+v^2)} \qquad (23)$$

erected on the uv plane. $u^{2m-1}v^{2n-1}e^{-(u^2+v^2)}du\, dv$ can be considered the volume of an elementary rectangular prism erected on the base du dv and having altitude z, as shown in Fig. 5. The integral written in Eq. (22) calls for the summation of the volumes of all these elementary prisms, by which process one finds the total volume under the surface generated by Eq. (23). The summation (integration) can be carried out by introducing polar coordinates in the uv plane. Let

$$u = r\cos\varphi, \quad v = r\sin\varphi.$$

Then the element of area du dv in Eq. (22) is to replaced by $r\, dr\, d\varphi$. A quadrant of the surface described by Eq. (23) can be mapped out by allowing φ to vary from 0 to $\frac{1}{2}\pi$ and r from 0 to ∞. That the integral in Eq. (22) actually exists (i.e., approaches a definite value as the upper limits for u and v are made larger and larger) can be demonstrated by showing that the volume under the surface of Eq. (23) outside the radius R can be reduced to any desired value however small simply by making R big enough. Eq. (22) then leads to the following equalities:

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty r^{2m-1}r^{2n-1}e^{-r^2}r\, dr\int_0^{\frac{1}{2}\pi}\sin^{2n-1}\varphi\cos^{2m-1}\varphi\, d\varphi$$

$$= 2\int_0^\infty r^{2(n+m)-1}e^{-r^2}dr \cdot 2\int_0^{\frac{1}{2}\pi}\sin^{2n-1}\varphi\cos^{2m-1}\varphi\, d\varphi$$

$$= \Gamma(n+m)B(n,m).$$

[15] E. B. Wilson, Advanced Calculus, Art. 143 (Ginn and Company, 1912).

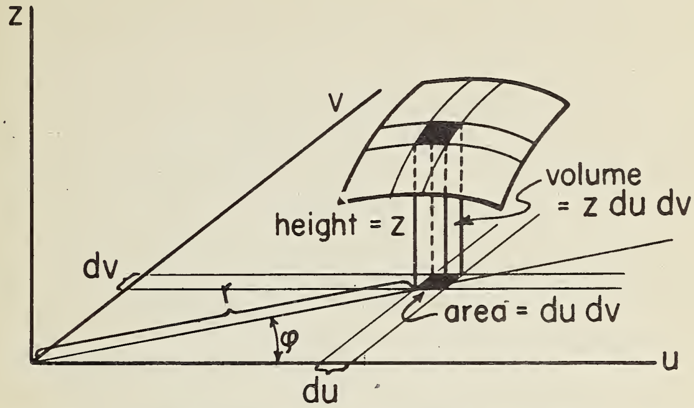

Fig. 5. The elementary prism of height z and base du dv, cut from the surface $z = u^{2m-1} v^{2n-1} e^{-(u^2+v^2)}$. The volume under the whole surface is the sum of the volumes of all such prisms.

by Eqs. (3) and (20). Thus we have discovered that

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \tag{24}$$

Another proof of this relation will be found in Exercise 11 at the end.

To find $\Gamma(\frac{1}{2})$.--At the close of the paragraph on other useful forms of the Beta function it was found that

$$B(\tfrac{1}{2},\tfrac{1}{2}) = \pi. \qquad \text{(See Eq. 21, p. 13)}$$

Now with $m = n = \frac{1}{2}$ in Eq. (24) we have

$$B(\tfrac{1}{2},\tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)}, \tag{25}$$

and since $\Gamma(1) = 1$ it follows that

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}. \tag{26}$$

Eq. (3) then gives

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-y^2}dy = \int_{-\infty}^\infty e^{-y^2}dy = \sqrt{\pi}. \tag{27}$$

The same result is derived in Exercise 25. By means of the integral just written, the student should satisfy himself

that the normal curve when written in the form

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \tag{28}$$

is normalized, i.e., that the area under the curve of y plotted against x is unity, and that it would not be so if the constant factor were other than $1/\sigma\sqrt{2\pi}$.

Tables of the incomplete Beta function.--The ratio $B_x(m,n)/B(m,n)$ is the fractional part of the total area under the curve $y = x^{m-1}(1-x)^{n-1}$ (Fig. 3) between 0 and the abscissa $x < 1$; and $1 - B_x(m,n)/B(m,n)$ or $B_{1-x}(n,m)/B(m,n)$ is the fractional part remaining between x and 1. This curve, which is the graph of the integrand of the Beta function, constitutes another important frequency curve, the Pearson Type I, and by change of variable, the Pearson Types II, VI, and VII also; hence the incomplete Beta function constitutes another important probability integral. The immense task of providing such tables was finally realized in 1934 with the publication by Karl Pearson and his staff of the Tables of the Incomplete Beta Function [16] --an even greater undertaking than the Tables of the Incomplete Gamma Function. A sample page is shown in Fig. 6. The function tabled is

$$I_x(p,q) = B_x(p,q)/B(p,q), \tag{29}$$

the arguments being p, q, and x. The introduction gives a history of the incomplete Beta function and illustrations of its use. The complete Beta function $B(p,q)$ is listed in the headings of the columns, so that if desired, $B_x(p,q)$ can be found by multiplying $I_x(p,q)$ by $B(p,q)$.

Besides supplying tables for integrals under certain frequency curves, the incomplete Beta function sums terms of the point binomial without approximation, as is demonstrated in a later section (p. 18).

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[16] Tables of the Incomplete Beta Function, issued by the Office of Biometrika, University College, Gower Street, London W.C. 1. Mention should be made of Tracts for Computers No. 7, by H. E. Soper, entitled The Numerical Evaluation of the Incomplete Beta Function, which will be found useful in the absence of, or outside the range of, the Tables of the Incomplete Beta Function.

128 TABLES OF THE INCOMPLETE β-FUNCTION

$x = \cdot 02$ to $\cdot 60$ $q = 5$ $p = 5$ to $7\cdot 5$

	$p = 5$	$p = 5\cdot 5$	$p = 6$	$p = 6\cdot 5$	$p = 7$	$p = 7\cdot 5$
$B(p, q) =$	$\cdot 1587\ 3016 \times \frac{1}{10^3}$	$\cdot 1108\ 4890 \times \frac{1}{10^3}$	$\cdot 7936\ 5079 \times \frac{1}{10^3}$	$\cdot 5806\ 3711 \times \frac{1}{10^3}$	$\cdot 4329\ 0043 \times \frac{1}{10^3}$	$\cdot 3281\ 8619 \times \frac{1}{10^3}$
x						
·02	·0000 004	·0000 001				
·03	·0000 028	·0000 006	·0000 001			
·04	·0000 113	·0000 029	·0000 007	·0000 002		
·05	·0000 332	·0000 096	·0000 028	·0000 008	·0000 002	·0000 001
·06	·0000 798	·0000 254	·0000 079	·0000 024	·0000 007	·0000 002
·07	·0001 666	·0000 572	·0000 193	·0000 064	·0000 021	·0000 007
·08	·0003 136	·0001 149	·0000 415⁻	·0000 147	·0000 052	·0000 018
·09	·0005 453	·0002 119	·0000 810	·0000 305⁺	·0000 114	·0000 042
·10	·0008 909	·0003 646	·0001 469	·0000 584	·0000 229	·0000 089
·11	·0013 838	·0005 936	·0002 507	·0001 044	·0000 429	·0000 175⁻
·12	·0020 615⁻	·0009 230	·0004 069	·0001 769	·0000 760	·0000 323
·13	·0029 649	·0013 808	·0006 332	·0002 864	·0001 279	·0000 565⁺
·14	·0041 384	·0019 986	·0009 505⁻	·0004 459	·0002 066	·0000 947
·15	·0056 287	·0028 117	·0013 832	·0006 713	·0003 219	·0001 527
·16	·0074 847	·0038 587	·0019 593	·0009 815⁺	·0004 858	·0002 379
·17	·0097 568	·0051 808	·0027 098	·0013 985⁻	·0007 131	·0003 598
·18	·0124 962	·0068 224	·0036 694	·0019 475⁻	·0010 214	·0005 300
·19	·0157 541	·0088 297	·0048 757	·0026 570	·0014 309	·0007 625⁻
·20	·0195 814	·0112 506	·0063 694	·0035 589	·0019 654	·0010 739
·21	·0240 280	·0141 343	·0081 935⁺	·0046 883	·0026 515⁻	·0014 839
·22	·0291 417	·0175 304	·0103 936	·0060 831	·0035 193	·0020 149
·23	·0349 682	·0214 888	·0130 167	·0077 843	·0046 020	·0026 926
·24	·0415 503	·0260 588	·0161 116	·0098 356	·0059 361	·0035 460
·25	·0489 273	·0312 883	·0197 277	·0122 827	·0075 612	·0046 073
·26	·0571 345⁻	·0372 238	·0239 148	·0151 734	·0095 196	·0059 122
·27	·0662 028	·0439 094	·0287 224	·0185 569	·0118 563	·0074 993
·28	·0761 583	·0513 861	·0341 994	·0224 834	·0146 187	·0094 105⁺
·29	·0870 218	·0596 916	·0403 932	·0270 037	·0178 560	·0116 907
·30	·0988 087	·0688 598	·0473 490	·0321 685⁻	·0216 192	·0143 873

Fig. 6. A sample page in the Tables of the Incomplete Beta Function.

To sum any number of terms of the point binomial without approximation.--Let S denote the sum of the terms of the point binomial $(q + p)^n$ as far as the term in p^t, which is to define

$$S = q^n + nq^{n-1} p + \binom{n}{2} q^{n-2} p^2 + \ldots + \binom{n}{t} q^{n-t} p^t$$

$$= \sum_{r=0}^{t} \binom{n}{r} q^{n-r} p^r = \frac{n!}{(n-r)!r!} q^{n-r} p^r \tag{30}$$

q and p are to be complementary; i.e., $q = 1 - p$, whence $dq/dp = -1$. Then if S be differentiated with respect to p, the result is

$$\frac{dS}{dp} = -\sum_{0}^{t} \frac{n!}{(n-r-1)!r!} q^{n-r-1} p^r + \sum_{1}^{t} \frac{n!}{(n-r)!(r-1)!} q^{n-r} p^{r-1}$$

$$= \qquad '' \qquad + \sum_{s=0}^{t-1} \frac{n!}{(n-s-1)!s!} q^{n-s-1} p^s$$

$$= -\frac{n!}{(n-t-1)!t!} q^{n-t-1} p^t. \tag{31}$$

This simple result is produced by the cancellation of plus and minus terms in the two summations; for every term in the first sum, except for $r = t$, there is a term of opposite sign in the second sum.

Now when $p = 0$, $S = 1$, whence from the last equation,

$$\int_1^S dS = -\frac{n!}{(n-t-1)!t!} \int_0^p x^t (1 - x)^{n-t-1} \, dx \tag{32}$$

which gives

$$S = 1 - \frac{n!}{(n-t-1)!t!} \int_0^p x^t (1 - x)^{n-t-1} \, dx. \tag{33}$$

Moreover, when $p = 1$, $q = 0$ and $S = 0$ if $t < n$, hence the factorials must satisfy the relation

$$\int_0^1 x^t (1 - x)^{n-t-1} \, dx = \frac{(n-t-1)!t!}{n!}. \tag{34}$$

This relation is equivalent to Eq. (24), but the evaluation of the factorials in terms of the Beta function comes in here as a by-product. It follows at once that

$$S = 1 - \frac{\int_0^p x^t (1 - x)^{n-t-1}\, dx}{\int_0^1 x^t (1 - x)^{n-t-1}\, dx} \tag{35}$$

Thus S turns out to be unity diminished by the ratio of an incomplete to a complete Beta function, and no approximation has been introduced. Unfortunately, the use of the incomplete Beta function, even with tables, is often laborious. Indeed, Eq. (35) can and has often been used as a means of evaluating the integral $\int_0^p x^t (1-x)^{n-t-1}\, dx$ by summing terms of the binomial; such was in fact the use that Bayes made of it in 1763 (Trans. Royal Society for the year 1763, p. 396). Laplace later derived the same result with the opposite purpose, viz., to sum terms of the binomial (<u>Théorie analytique</u>, 1823; p. 151).

The derivation followed here was published by Lazarus[17]; it is simpler than any of the others that I have seen.

<u>To sum any number of terms of the Poisson exponential limit without approximation.</u>--Let S denote the sum of the terms of the Poisson exponential limit as far as the term in m^t. Then

$$S = e^{-m}\{1 + m + m^2/2! + \ldots + m^t/t!\}. \tag{36}$$

Differentiation with respect to m gives

$$\frac{dS}{dm} = -S + e^{-m}\{1 + 2m/2! + 3m^2/3! + \ldots + t\, m^{t-1}/t!\}$$

$$= -e^{-m} m^t/t! \tag{37}$$

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[17]This method of deriving Eq. (35) was first published by William Lazarus of Hamburg in the Journal of the Institute of Actuaries, vol. 15, 1870; pp. 245-257; page 251 in particular. Lazarus credits this derivation to one Dr. Landi of Trieste. The result contained in Eq. (35) was first found by Bayes; Trans. Royal Soc. for the year 1763, page 396. This paper is available in a booklet entitled <u>Facsimiles of Two Papers by Bayes</u>, with commentaries by E. C. Molina and W. Edwards Deming, published by the Graduate School of the Department of Agriculture, 1940.

all the terms in the braces, except for the last, being cancelled by opposite terms in S. Now when m = 0, S = 1, so

$$\int_1^S dS = (1/t!) \int_0^m x^t e^{-x}\, dx$$

$$S = 1 - (1/t!) \int_0^m x^t e^{-x}\, dx. \tag{38}$$

Moreoever, when $m = \infty$, S = 0, and it follows that

$$\int_0^\infty x^t e^{-x}\, dx = t! \tag{39}$$

This relation is equivalent to Eq. (8), but the identification of the complete Gamma function with the factorial comes in here as a by-product. It follows at once that

$$S = 1 - \int_0^m x^t e^{-x}\, dx \Big/ \int_0^1 x^t e^{-x}\, dx. \tag{40}$$

Thus, the incomplete sum of the Poisson exponential limit is expressible in terms of an incomplete Gamma function. Turned around, the incomplete Gamma function is expressible as a sum of terms of the Poisson limit. Eq. (40) was first published by E. C. Molina in 1915 (Amer. Math. Monthly, vol. 22, p. 223, 1915). It is usually derived by integration by parts, but Lazarus' method seems simpler.

EXERCISES

1. Plot the curve $y = x^{n-1} e^{-x}$ for n = 1, 2, 3, 6, 10 between x = 0 and x = 8. The areas under these curves to infinity are respectively $\Gamma(1) = 1$, $\Gamma(2) = 1$, $\Gamma(3) = 2$, $\Gamma(4) = 3! = 6$. Notice that as $x \to \infty$ the slopes of all the curves become zero. If n = 1 the curve starts from the point (0,1) with slope -1; if n = 2 the curve starts from the origin with slope +1. If n > 2 the curve starts from the origin with a horizontal tangent, making contact of order n - 2. At infinity all the curves make contact of infinitely high order with the x axis.

2. (a) Plot the curve $y = x^{m-1}(1-x)^{n-1}$ for m, n = 1, 2; 2, 1; 2, 3; 3, 2; $\frac{1}{2}$, 2; 2, $\frac{1}{2}$ between x = 0 and x = 1.

(b) Determine the numerical values of the areas under these curves, and check roughly with the graphs. Notice that when $n = m$ the curves are symmetrical, and when $n \neq m$ the curves are lopsided; if $n < m$ the bulk of the area is thrown to the right, and if $n > m$ the bulk of the area is thrown to the left. The order of contact with the x axis is $m - 2$ at $x = 0$ and $n - 2$ at $x = 1$.

3. Given $\Gamma(\frac{1}{2}) = \sqrt{\pi}$; find by the recursion formula the values of $\Gamma(\frac{3}{2})$, $\Gamma(\frac{5}{2})$, $\Gamma(\frac{7}{2})$, $\Gamma(\frac{9}{2})$. Show that $(n + \frac{1}{2})!$ $= \Gamma(n + \frac{3}{2}) = \pi^{\frac{1}{2}} \dfrac{1 \cdot 3 \cdot 5 \ldots (2n + 1)}{2^{n+1}}$.

4. By the integral $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$, show that $\Gamma(n)$ increases without limit as $n \to 0$ from the positive side. Proof: Near the origin, e^{-x} can be replaced by 1, so that the integrand becomes x^{n-1}. The integral $\int_0^a x^{n-1} dx$, $0 < a$, is finite as long as $0 < n$; but no matter how small a is, this integral can be made greater than any preassigned number however large by making n small enough. In other words the integral approaches $+\infty$ as $n \to 0$ from the positive side. Hence

$$\lim_{n \to +0} \Gamma(n) = +\infty$$

(See Fig. 2.)

5. (a) By the recursion formula show that $\Gamma(-\frac{1}{2}) = -2\Gamma(\frac{1}{2}) = -2\sqrt{\pi}$.

(b) $\Gamma(-n + \frac{1}{2}) = \sqrt{\pi}(-2)^n / 1.3.5 \ldots (2n - 1)$.

Notice that $\Gamma(-n + \frac{1}{2})$ alternates in sign and decreases in magnitude as n increases. This formula evaluates the Gamma function at the negative half-integers; see Fig. 2.

6. Show by the recursion formula that $\Gamma(-\frac{1}{4}) = 4\Gamma(\frac{3}{4})$; $\Gamma(-\frac{1}{8}) = -8\Gamma(\frac{7}{8})$, $\Gamma(-\frac{1}{32}) = -32\Gamma(\frac{31}{32})$, ...; $\Gamma(-\frac{3}{4}) = -(\frac{4}{3})\Gamma(\frac{1}{4})$, $\Gamma(-\frac{7}{8}) = -(\frac{8}{7})\Gamma(\frac{1}{8})$, $\Gamma(-\frac{31}{32}) = -(\frac{32}{31})\Gamma(\frac{1}{32})$, ... So $\Gamma(n)$ approaches $-\infty$ as n approaches 0 from the negative side or -1 from the positive side.

7. Show that if $n!$ be identified with $\Gamma(n+1)$, then $0! = 1$.

8. Show that

$$\int_0^\infty e^{-bx-a^2x^2}\,dx = \frac{e^{b^2/4a^2}}{2|a|}[\sqrt{\pi} - \Gamma_{b^2/4a^2}(\tfrac{1}{2})],\quad b \geqq 0$$

$$\int_0^\infty e^{bx-a^2x^2}\,dx = \frac{e^{b^2/4a^2}}{2|a|}[\sqrt{\pi} + \Gamma_{b^2/4a^2}(\tfrac{1}{2})],\quad b \geqq 0$$

By addition

$$2\int_0^\infty e^{-a^2x^2}\cosh bx\,dx = \int_0^\infty e^{bx-a^2x^2}\,dx = \frac{e^{b^2/4a^2}}{|a|}\sqrt{\pi},\quad b \lesseqgtr 0.$$

Given: $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

9. $\int_0^\infty t^{n-1}e^{-t^n}dt = \int_0^\infty t^{2n-1}\,e^{-t^n}\,dt = \frac{1}{n}.$

$\int_0^\infty t^{m-1}e^{-t^n}\,dt = (\frac{1}{n})\Gamma(\frac{m}{n})$, which becomes $\pi^{\frac{1}{2}}/n$ if $n = 2m$.

(Kramp, Analyse des réfractions, Strasbourg, 1799).

10.
$$\Gamma(\tfrac{n-1}{2}) = \begin{cases} \frac{n-3}{2}\cdot\frac{n-5}{2}\cdot\frac{n-7}{2}\cdots\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} & n \text{ even} \\ \frac{n-3}{2}\cdot\frac{n-5}{2}\cdot\frac{n-7}{2}\cdots\frac{6}{2}\cdot\frac{4}{2}\cdot\frac{2}{2} & n \text{ odd} \end{cases}$$

$$= \begin{cases} \sqrt{\pi}\,\dfrac{(n-2)!}{2^{n-2}(\frac{n-2}{2})!} & n \text{ even} \\ \left[\tfrac{1}{2}(n-3)\right]! & n \text{ odd} \end{cases}$$

By means of these verify your answers for Ex. 3.

11. By trigonometric substitution, show that the Euler integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ defining $B(m,n)$ is equivalent to $2\int_0^{\frac{1}{2}\pi}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$. Integrate this by Wallis' formula and show that

$$B(m,n) = B(n,m) = \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

which is Eq. (24).

12. $$\frac{1}{B(\frac{n-1}{2},\ \frac{1}{2})} = \frac{\Gamma(\frac{1}{2}n)}{\Gamma(\frac{n-1}{2})\,\Gamma(\frac{1}{2})}$$

$$\left.\begin{aligned} &= \frac{1}{\pi}\,\frac{n-2}{n-3}\,\frac{n-4}{n-5}\,\frac{n-6}{n-7}\cdots\frac{6\cdot 4\cdot 2}{5\cdot 3\cdot 1} = \frac{2^{n-2}(\frac{n-2}{2}!)^2}{\pi(n-2)!} \quad n \text{ even} \\ &= \frac{1}{2}\,\frac{n-2}{n-3}\,\frac{n-4}{n-5}\,\frac{n-6}{n-7}\cdots\frac{5\cdot 3\cdot 1}{4\cdot 2} = \frac{(n-2)!}{2^{n-2}(\frac{n-3}{2}!)^2} \quad n \text{ odd} \end{aligned}\right\}$$

13. The number of possible combinations of n articles taken r at a time is $n!/(n-r)!\,r!$, which is often abbreviated $\binom{n}{r}$, as in Eq. (30).

(a) Show that if the factorials in this expression be identified with Gamma functions, then $\binom{n}{r}$ vanishes when r is any negative integer or any positive integer greater than n (a fact already mentioned in the text).

(b) Show that $\binom{n}{n} = \binom{n}{0} = 1$.

(c) Show that $\Sigma\, r\binom{n}{r}p^r q^{n-r} = np$ where the sum is to be taken over all integral values of r.

14. Show that $\sum\limits_{r=0}^{n} \dfrac{1}{B(n-r+1,\ r+1)} = 2^n(n+1)$, or that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$. Hint: Expand $(\frac{1}{2} + \frac{1}{2})^n$ by the binomial theorem and note that the sum of the terms is unity. The summation could as well be taken over all integral values of r, positive and negative; see the preceding exercise.

15. Show that $B(m,n) = \int_0^1 \dfrac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx$. Hint: replace x in $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ by $z/(1+z)$ and then by $1/(1+z)$ and get $B(m,n) = \int_0^\infty z^{m-1}(1+z)^{-m-n}\,dz$ and $B(m,n)$ $= \int_0^\infty z^{n-1}(1+z)^{-m-n}\,dz$. Add these and obtain $B(m,n) =$

$$\frac{1}{2}\int_0^\infty \frac{z^{m-1} + z^{n-1}}{(1+z)^{m+n}}\,dz = \frac{1}{2}\int_0^1 \frac{z^{m-1} + z^{n-1}}{(1+z)^{m+n}}\,dz + \frac{1}{2}\int_1^\infty \frac{z^{m-1} + z^{n-1}}{(1+z)^{m+n}}\,dz$$

$$= \frac{1}{2}\int_0^1 \frac{z^{m-1} + z^{n-1}}{(1+z)^{m+n}}\,dz + \frac{1}{2}\int_0^1 \frac{v^{n-1} + v^{m-1}}{(1+v)^{m+n}}\,dv \qquad \text{(z = 1/v in the last integral)}$$

$$= \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx.$$

16. Prove the following recursion formula for the incomplete Beta function--

$$I_x(p,q) = xI_x(p-1,q) + (1-x)\, I_x(p,q-1)$$

Here, as in the <u>Tables of the Incomplete Beta Function</u>,

$$I_x(p,q) \equiv B_x(p,q)/B(p,q) = \frac{1}{B(p,q)} \int_0^x x^{p-1}(1-x)^{q-1}\,dx.$$

Solution due to Mr. Martin Katzin,

Naval Research Laboratory, April 1942.

The problem is to prove the recursion formula

$$I_x(p,q) = xI_x(p-1,q) + (1-x)I_x(p,q-1) \tag{1}$$

Start with

$$I_x(p,q) = \frac{B_x(p,q)}{B(p,q)} = \frac{1}{B(p,q)} \int_0^x t^{p-1}(1-t)^{q-1}\,dt \tag{2}$$

then write

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \frac{(p-1)\Gamma(p-1)\Gamma(q)}{(p+q-1)\Gamma(p+q-1)}$$

$$= \frac{q-1}{p+q-1}\, B(p,q-1)$$

$$= \frac{q-1}{p+q-1}\, B(p,q-1) \tag{3}$$

whence

$$(p+q-1)\, B(p,q) = (p-1)\, B(p-1,q) = (q-1)\, B(p,q-1) \tag{4}$$

Also

$$B_x(p-1,q) = \int_0^x t^{p-2} (1-t)^{q-1}\, dt$$

$$= \left[\frac{t^{p-1}}{p-1}(1-t)^{q-1} \right]_0^x + \frac{q-1}{p-1} \int_0^x t^{p-1} (1-t)^{q-2}\, dt$$

$$= \frac{x^{p-1}(1-x)^{q-1}}{p-1} + \frac{q-1}{p-1} B_x(p,q-1) \tag{5}$$

whence

$$(p-1)\, B_x(p-1,q) = (q-1)\, B_x(p,q-1) + x^{p-1} (1-x)^{q-1} \tag{6}$$

Divide these terms by the three members of Eq. (4) and get

$$I_x(p-1,q) - I_x(p,q-1) = \frac{x^{p-1} (1-x)^{q-1}}{(p+q-1)B(p,q)} \tag{7}$$

Now work with $B_x(p,q)$. We need something different from any reduction yet accomplished. Find now that

$$B_x(p,q) = \int_0^x t^{p-1} (1-t)^{q-1}\, dt$$

$$= \left[\frac{t^p}{p} (1-t)^{q-1}\right]_0^x + \frac{q-1}{p}\int_0^x t^p(1-t)^{q-1}\, dt \tag{8}$$

whence

$$\frac{x^p(1-x)^{q-1}}{p} = B_x(p,q) - \frac{q-1}{p}\int_0^x t^p(1-t)^{q-2}\, dt \tag{9}$$

It is of course true that

$$-(q-1)\int_0^x t^p(1-t)^{q-2}\, dt = +\, (q-1)\int_0^x -t \cdot t^{p-1}(1-t)^{q-2}\, dt$$

$$= (q-1) \int_0^x (1-t-1)\, t^{p-1}(1-t)^{q-2}\, dt$$

$$= (q-1) \int_0^x t^{p-1}(1-t)^{q-1}\, dt - (q-1) \int_0^x t^{p-1}(1-t)^{q-2}\, dt$$

$$= (q-1)\, B_x(p,q) - (q-1)\, B_x(p,q-1) \tag{10}$$

So

$$x^p(1-x)^{q-1} = p\, B_x(p,q) + (q-1)\, B_x(p,q) - (q-1)\, B_x(p,q-1) \tag{11}$$

or

$$\frac{x^{p}(1-x)^{q-1}}{(p+q-1)\ B(p,q)} = \frac{B_x(p,q)}{B(p,q)} - \frac{q-1}{p+q-1}\ \frac{B_x(p,q-1)}{B(p,q)}$$

$$= I_x(p,q) - I_x(p,q-1) \quad \text{(by Eq. 4)} \tag{12}$$

Also, from Eq. (7) we have

$$\frac{x \cdot x^{p-1}\ (1-x)^{q-1}}{(p+q-1)\ B(p,q)} = x[I_x(p-1,q) - I_x(p,q-1)]$$

and by comparison it is seen that

$$I_x(p,q) = x\ I_x(p-1,q) + (1-x)\ I_x(p,q-1), \text{ Q.E.D.}$$

17. In the analysis of variance occurs the frequency curve

$$y\ dw = \frac{2\ k'^{\frac{1}{2}k'}\ k''^{\frac{1}{2}k''}}{B(\frac{1}{2}k',\ \frac{1}{2}k'')}\ \frac{w^{k'-1}}{(k'' + k'w^2)^{\frac{1}{2}(k'+k'')}}\ dw$$

for the distribution of w, which stands for the ratio of two estimates of σ. By transforming to x, show that if P(w) denotes the fractional part of the area lying beyond a given abscissa w, then

$$P(w) = 1 - \frac{1}{B(p,q)} \int_0^x x^{p-1}(1-x)^{q-1}\ dx$$

$$= 1 - I_x(p,q) = I_{1-x}(q,p)$$

wherein x stands for $k'w^2/(k''+k'w^2)$, and $p = \frac{1}{2}k'$, $q = \frac{1}{2}k''$ Thus the evaluation of P(w) can be made to depend on an incomplete Beta function.

18. The Type III curve is sometimes written $y = y_0(1 + x/a)^{p}\ e^{-px/a}$ in order to put the zero of x at the mode (maximum). Select a point B lying at the distance b from the finite end of the curve, and show that the fractional part of the area lying beyond the point B is

$$\int_{bp/a}^{\infty} v^{p}\ e^{-v}\ dv \Big/ \int_0^{\infty} v^{p}\ e^{-v}\ dv = 1 - I(u,p)$$

if $u = bp/a(1+p)^{\frac{1}{2}}$ in the notation of the Tables of the Incomplete Gamma Function (p. 9).

19. Prove that $\int_0^t x^{m-1}(a-x)^{n-1}\,dx = a^{m+n-1} B_{t/a}(m,n)$, $a > 0$.

20. The volume bounded by the xy plane, the plane $x+y = 1$, and the surface $z = x^{\ell-1} y^{m-1}$, is $\Gamma(\ell)\,\Gamma(m)/\Gamma(\ell + m + 1)$.

21. (a) The integral $\iiint x^{\ell-1} y^{m-1} z^{n-1}\,dx\,dy\,dz$ taken over all positive values of x, y, z such that $x+y+z \leqq 1$ is equal to $\Gamma(\ell)\,\Gamma(m)\,\Gamma(n)/\Gamma(\ell + m + n + 1)$.

(b) The integral $\iiint x^{\ell-1} y^{m-1} z^{n-1}\,dx\,dy\,dz$ taken over all positive values of x, y, z such that $x/a + y/b + z/c \leqq h$, is equal to $a^{\ell} b^{m} c^{n} h^{\ell+m+n}\,\Gamma(\ell)\,\Gamma(m)\,\Gamma(n)/\Gamma(\ell + m + n + 1)$. Hint: Transform the integral of part (a) by putting $x, y, z = ah\xi, bh\eta, ch\zeta$.

(c) The integral $\iiint x^{\ell-1} y^{m-1} z^{n-1}\,dx\,dy\,dz$ taken over all values of x, y, z lying within the positive octant of the surface $(x/a)^{p} + (y/b)^{q} + (z/c)^{r} \leqq h$ is equal to

$$h^{\ell/p + m/q + n/r} \times \frac{a^{\ell} b^{m} c^{n}}{pqr} \times \frac{\Gamma(\ell/p)\,\Gamma(m/q)\,\Gamma(n/r)}{\Gamma(\ell/p + m/q + n/r + 1)}.$$

Hint: Transform to an integral like that in part (a) by putting $\xi h, \eta h, \zeta h = (x/a)^{p}, (y/b)^{q}, (z/c)^{r}$.

(d) By the results of part (c) show that the volume of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ is $\frac{4}{3}\pi abc$ and hence that the volume of a sphere is $\frac{4}{3}\pi r^3$. [Hint: Put $h = 1$; $p = q = r = 2$; $\ell = m = n = 1$.]

(e) By a similar artifice, show that the volume of the hypocycloid $(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} = 1$ is $4\pi\, abc/35$.

(f) The center of gravity of the positive octant of the ellipsoid of part (d) lies at $x, y, z = 3a/8$,

3b/8, 3c/8. Hint: In part (c) put $\ell = 2$, $m = n = 1$, $h = 1$, $p = q = r = 2$ to find the x coordinate of the center of gravity.

22. (a) The n-fold integral $\iint \ldots \int x_1^{m_1-1} x_2^{m_2-1} \ldots x_n^{m_n-1}\, dx_1 dx_2 \ldots dx_n$ taken over all positive values of x_1, x_2, ..., x_n such that $x_1 + x_2 + \ldots x_n + \leqq 1$, is equal to $\Gamma(m_1)\Gamma(m_2)\ldots \Gamma(m_n)/\Gamma(m_1 + m_2 + \ldots + m_n + 1)$. [This is an easy extension of part (a) of the preceding exercise.]

(b) The volume of the n-dimensional ellipsoid $(x_1/a_1)^2 + (x_2/a_2)^2 + \ldots + (x_n/a_n)^2 = 1$ is $V = a_1 a_2 \ldots a_n \dfrac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n+1)}$, and the volume of the n-dimensional sphere $x_1^2 + x_2^2 + \ldots + x_n^2 = r^2$ is $\dfrac{\pi^{\frac{1}{2}n} r^n}{\Gamma(\frac{1}{2}n+1)}$. Hint: The volume V of the n-dimensional ellipsoid is by definition 2^n times the integral of $dx_1 dx_2 \ldots dx_n$ taken over all positive values of x_1, x_2, ..., x_n such that $(x_1/a_1)^2 + (x_2/a_2)^2 + \ldots + (x_n/a_n)^2 \leqq 1$. By the change of variable $(x_i/a_i)^2 = \xi_i$ and $dx_i = \frac{1}{2}a_i \xi_i^{-\frac{1}{2}} d\xi_i$, the volume V is seen to be 2^n times the integral of $\dfrac{a_1 a_2 \ldots a_n}{2^n(\xi_1 \xi_2 \ldots \xi_n)^{\frac{1}{2}}}\, d\xi_1 d\xi_2 \ldots d\xi_n$ taken over all positive values of ξ_1, ξ_2, ..., ξ_n such that $\xi_1 + \xi_2 + \ldots + \xi_n \leqq 1$. The result of part (a) with $m_1 = m_2 = \ldots = m_n = \frac{1}{2}$ applies at once, giving $V = a_1 a_2 \ldots a_n \dfrac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n+1)}$ as written above.

23. (a) The volume of the thin spherical shell lying between the pair of n-dimensional spheres $x_1^2 + x_2^2 + \ldots + x_n^2 = (r \pm \frac{1}{2}dr)^2$ is $\dfrac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)} r^{n-1}dr$. [Hint: simply differentiate the volume of the sphere just found.] This result was used by Helmert in developing the distribution of the standard deviations in samples of n drawn from a normal population. See Czuber's Beobachtungsfehler (Teubner, 1891) on pages 147-150. The student should observe that this result reduces to $4\pi r^2\, dr$ for the volume of a

spherical shell in three dimensions, and to $2\pi r\,dr$ for the area of a circular ring in two dimensions.

(b) The volume of the thin shell lying between the pair of n-dimensional ellipsoids $(x_1/a_1)^2 + (x_2/a_2)^2 + \ldots + (x_n/a_n)^2 = (s \pm \frac{1}{2}ds)^2$ is $a_1 a_2 \ldots a_n \dfrac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)} s^{n-1}ds$.

This result is useful in developing the distribution of the standard deviations in samples of n drawn from a normal population. See page 126 in Deming and Birge's Statistical Theory of Errors (The Graduate School, Department of Agriculture).

24. The area of a sphere of radius r is $\{2\pi^{\frac{1}{2}n}/\Gamma(\frac{1}{2}n)\}r^{n-1}$. Hint: simply take off the dr from the result of Exercise 23. The student should observe that in three dimensions this result reduces to $4\pi r^2$ for the area of a sphere, and in two dimensions to $2\pi r$ for the circumference of a circle.

25. Euler in 1768 showed that

$$\int_0^\infty \frac{y^{n-1}}{1+y}\,dy = \pi/\sin n\pi \quad \text{if } 0 < n < 1.$$

Use this result, and put $x = y/(1+y)$ in Eq. (13) to prove that

(a) $\Gamma(n)\Gamma(1-n) = B(n,1-n) = \pi/\sin n\pi$

if $0 < n < 1$.

(b) Set $n = \frac{1}{2}$ and prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, whence

$$\int_0^\infty e^{-x^2}\,dx = \tfrac{1}{2}\sqrt{\pi}.$$

(c) Prove that $\Gamma(\frac{1}{2}) = \pi/\Gamma(\frac{1}{2})$

$\Gamma(\frac{3}{4}) = \sqrt{2}\,\pi/\Gamma(\frac{1}{4})$

$\Gamma(\frac{5}{6}) = 2\pi/\Gamma(\frac{1}{6})$

Etc.

(d) Prove that $\Gamma(1/n)\Gamma(2/n)\Gamma(3/n) \ldots \Gamma(\overline{n-1}/n) = (2\pi)^{\frac{1}{2}(n-1)}/\sqrt{n}$.

Hint: Set A equal to the left-hand side. Then

$$A^2 = \Gamma(1/n)\Gamma(\overline{n-1}/n) \cdot \Gamma(2/n)\Gamma(\overline{n-2}/n) \cdot \ldots \cdot \Gamma(\overline{n-1}/n)\Gamma(1/n)$$
$$= \{\pi/\sin \pi/n\}\{\pi/\sin 2\pi/n\} \ldots \{\pi/\sin \overline{n-1}\ \pi/n\}$$

by part (a). Then let $\theta \to 0$ in the following trigonometric identity (Hobson: Plane Trigonometry)

$$\sin n\theta/\sin \theta = 2^{n-1}\sin(\theta + \pi/n)\sin(\theta + 2\pi/n) \ldots \sin(\theta + \overline{n-1}\ \pi/n)$$

with the result that $A^2 = (2\pi)^{n-1}/n$, which is equivalent to the theorem stated.

26. Show that the relation between the Gamma and the Beta functions (Eq. 24) can be found without recourse to integration in polar coordinates.

Hint: (Called to my attention by Mr. Arnold Frank) Evaluate

$$I = \int_0^\infty \int_0^\infty e^{-x(y+1)} x^{m+n-1} y^{n-1}\, dx\, dy,$$

first with respect to x, and then with respect to y, to find

$$I = \Gamma(m+n)B(m,n) \qquad \{\text{Let } z = x(y+1)\}.$$

Now evaluate the same integral, first with respect to y, and then with respect to x to find

$$I = \Gamma(m)\Gamma(n) \qquad \{\text{Let } z = xy\}.$$

The order of integration is immaterial, whereupon these two results can be set equal, and the relation between the Gamma and the Beta functions follows. That the order of integration is immaterial can be justified by a theorem due to de la Vallée Poussin. This theorem states that the values of certain infinite multiple integrals are independent of the order of integration.

27. Expansion of the incomplete Gamma function in series.

(a) Expand e^{-x} in powers of x and get

$$\Gamma_x(n) = \int_0^x x^{n-1}e^{-x}dx$$

$$= \{x^n/n\}\{1 - \frac{n}{n+1}x + \frac{n}{n+2}x^2 - \frac{n}{n+3}x^3 + \ldots\}.$$

This series is convergent for all values of x, but is not convenient for numerical calculation unless x is small.

(b) By integrating by parts show that

$$\int_0^x x^{n-1}e^{-x}dx = \{x^n e^{-x}/n\}\{1 + x/(n+1) + x^2/(n+1)(n+2)$$

$$+ x^3/(n+1)(n+2)(n+3) + \ldots\}.$$

This series also converges for all values of x, but is quicker calculated for moderate values of x than the preceding series, especially if n is large.

(c) Again, by integrating by parts show that

$$\int_0^x x^{n-1}e^{-x}dx = \Gamma(n) - x^{n-1}e^{-x}\{1 + (n-1)/x + (n-1)(n-2)/x^2$$

$$+ (n-1)(n-2)(n-3)/x^3 + \ldots\}.$$

If n is an integer, this series will terminate. If n is not an integer, the series will be infinite and moreover will diverge for any x no matter how large. This can be seen from the ratio test, since the ratio of the s-th term to (s-1)th term is (n-s)/x. Since n > 0, the series will eventually oscillate, and oscillate infinitely for any x. But no matter where curtailed, the error committed will be less than the next term (Laplace Théorie analytique, pp. 174-5 in the 3d ed., 1820). There will always be an optimum term (Bayes, 1763), and the series carried just short of the optimum term will give the best possible representation for those values of x and n. When x is large, a very few terms will suffice for accurate calculations, especially if n is not large.

> Note: Series (b) and (c) can also be derived by the D operator used inversely, where $D = d/dx$ and $1/D = \int$.

28. Show that $\int_0^x x^n e^{-x}\,dx = x^n e^{-x} v$, where

$$v = \cfrac{1}{1 - ny\cfrac{1}{1 + y\cfrac{1}{1 - (n-1)y\cfrac{1}{1 + 2y\cfrac{1}{1 - (n-2)y\cfrac{1}{1 + 3y\cfrac{1}{1+\dots}}}}}}} \qquad (y = 1/x)$$

Hint:

1' Show that v satisfies the differential equation

$$x\,dv/dx = (x-n)v - x. \tag{1}$$

2' Show that a solution of the general Riccati equation

$$x\,dv/dx = (x-a)v - x + bv^2 \tag{2}$$

is
$$v = 1/(1 + kyv_1) = x/(x + kv_1), \quad (y = 1/x) \tag{3}$$

provided v_1 satisfies the differential equation

$$x\,dv_1/dx = (x + a + 1)v_1 - x(b-a)/k + kv_1^2. \tag{4}$$

3' This is a Riccati equation like (2) if $k = b - a$.
Hence a solution of (2) is

$$v = 1/(1 + [b-a]v_1 y) \tag{5}$$

wherein v_1 is such that it satisfies the differential equation .

$$x\,dv_1/dx = (x + a + 1) - x + (b-a)v_1^2. \tag{6}$$

4' It is evident, then, that

$$v_1 = 1/(1 + [b + 1]yv_2) \tag{7}$$

if v_2 satisfies

$$dv_2/dx = (x - a)v_2 - x + (b + 1)v_2^2 \tag{8}$$

This is like Eq. (2), but with b replaced by b + 1. Hence we may say that

$$\begin{aligned} y &= 1/(1 + [b - a]yv_1) \\ v_1 &= 1/(1 + [b + 1]yv_2) \\ v_2 &= 1/(1 + [b + 1 - a]yv_3) \\ v_3 &= 1/(1 + [b + 2]yv_4) \\ v_4 &= 1/(1 + [b + 2 - a]yv_5) \end{aligned}$$

etc., and the required result is established.

29. The mean of the curve $y = x^{n-1} e^{-x}$ (Fig. 1, p. 2) between 0 and ∞ is at n, its mode (maximum) is at n - 1, its second moment coefficient about the y axis is n(n + 1), and its standard deviation is $n^{\frac{1}{2}}$.

30. The mean of the curve $y = x^{m-1}(1 - x)^{n-1}$ lying between 0 and 1 is at m/(m + n), its mode is at (m - 1)/(m + n - 2), and its standard deviation is $(mn)^{\frac{1}{2}}/(m + n)(m + n + 1)^{\frac{1}{2}}$.

31. Show by Eq. (19) that if the area under the curve

$$y\,dz = C(1 + z^2)^{-\frac{1}{2}n}\,dz$$

from $-\infty$ to $+\infty$ is unity, i.e., if the curve is "normalized," C must have the value $1/B[\frac{1}{2}(n - 1),\frac{1}{2}]$. This curve is the distribution of Student's z, n being the sample size.

32. Prove that the standard deviation of the distribution of Student's z is $1/\sqrt{(n - 3)}$. (Student's distribution of z is the curve used in the preceding exercise.) What happens to the standard deviation of this curve if n is 3 or less?

33. Helmert's curve for the distribution of the standard deviation s in samples of n drawn from a normal universe is

$$y\,ds = C\,\left(\frac{s}{\sigma}\right)^{n-2} e^{-ns^2/2\sigma^2}\,ds$$

If the area under the curve from $s = 0$ to $s = \infty$ is unity, the value of C must be $\dfrac{n^{\frac{1}{2}(n-1)}}{\Gamma[\frac{1}{2}(n-1)]\;2^{\frac{1}{2}(n-3)}\;\sigma}$, which is the normalizing factor. The distribution of s was first derived by Helmert in 1876; see p. 126 in Deming and Birge's Statistical Theory of Errors, published by the Graduate School, Department of Agriculture.

> Hint: In this exercise and the next nine it will be found convenient to use the change of variable $ns^2/2\sigma^2 = x$ and $2\,ds/s = dx/x$, and to write Eq. (2) in the form
>
> $$\Gamma_x(n) = \int_0^x x^n\, e^{-x}\,\frac{dx}{x}$$

34. Show that if P(s) is defined by the equation

$$P(s) = \frac{\int_s^\infty \left(\frac{s}{\sigma}\right)^{n-2} e^{-ns^2/2\sigma^2}\,ds}{\int_0^\infty \left(\frac{s}{\sigma}\right)^{n-2} e^{-ns^2/2\sigma^2}\,ds}$$

then

$$P(s) = 1 - \frac{\Gamma_x[\frac{1}{2}(n-1)]}{\Gamma\,[\frac{1}{2}(n-1)]}$$

The integral P(s) is the area under the distribution of s from the abscissa s to infinity, and the evaluation of P(s) is here seen to depend on an incomplete Gamma function. See Exercise 42 for the identification of P(s) with P(χ) which is the corresponding integral under the distribution of χ.

35. Show that the mean of the curve in the preceding exercise is

$$Es = \sigma \sqrt{\frac{2\pi}{n}} \frac{1}{B[\frac{1}{2}(n-1), \frac{1}{2}]}$$

The symbol E denotes expectation or mathematical average. Es is the theoretical average standard deviation in samples of n drawn from a normal universe of standard deviation σ.

36. Show that for Helmert's distribution of s (Exercise 33),

$$Es^2 = \int_0^\infty s^2 \, y \, ds = \frac{n-1}{n} \sigma^2$$

Es^2 is the theoretical average of the square of the standard deviation in samples of n drawn from a normal universe or any other universe of standard deviation σ.

37. Helmert's curve for the distribution of the root-mean-square error δ in samples of n drawn from a normal universe is

$$y \, d\delta = C \left(\frac{\delta}{\sigma}\right)^{n-1} e^{-n\delta^2/2\sigma^2} \, d\delta$$

If the area under the curve from $\delta = 0$ to $\delta = \infty$ is unity, the value of C must be $\dfrac{n^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)\, 2^{\frac{1}{2}(n-2)}\, \sigma}$, which is the normalizing factor.

38. For the curve in the preceding exercise show that

$$E\,\delta = \int_0^\infty \delta y \, d\delta = \sigma \sqrt{\frac{2\pi}{n}} \frac{1}{B(\frac{1}{2}n, \frac{1}{2})}$$

$$E\,\delta^2 = \int_0^\infty \delta^2 \, y \, d\delta = \sigma^2$$

$E\delta$ is the theoretical average root-mean-square error in samples of n drawn from a normal universe, and $E\delta^2$ is the theoretical average mean square error for a normal universe or any other universe.

39. Without actually performing the integrations or reducing the integrals to Gamma functions, but rather by making use of the results in Exercises 36 and 38, show that the mean of the distribution of s^2 must be $\frac{n-1}{n}\sigma^2$ and that the mean of the distribution of δ^2 must be σ^2.

40. When there are k degrees of freedom the distribution of χ^2 is

$$y\, d\chi^2 = C\, (\chi^2)^{\frac{1}{2}(k-2)}\, e^{-\frac{1}{2}\chi^2}\, d\chi^2$$

If the area under the curve from $\chi^2 = 0$ to $\chi^2 = \infty$ is to be unity, show that C must have the value $\dfrac{1}{\Gamma(\frac{1}{2}k)\, 2^{\frac{1}{2}k}}$.

41. The mean of the distribution of χ^2 is at k and its standard deviation is $\sqrt{2k}$.

42. With P(s) defined as in Exercise 33 and with

$$P(\chi) = \frac{1}{\Gamma(\frac{1}{2}k)\, 2^{\frac{1}{2}k}} \int_{\chi^2}^{\infty} (\chi^2)^{\frac{1}{2}(k-2)}\, e^{-\frac{1}{2}\chi^2}\, d\chi^2$$

show that if $ns^2/\sigma^2 = \chi^2$ and $n - 1 = k$, then $P(s) = P(\chi)$. The results of this example show that instead of finding P(s) from the Tables of the Incomplete Gamma Function, P(s) can be found from tables of χ^2 which give $P(\chi)$ as a function of χ^2 and k.

43. Show that the value of C required to normalize the curve

$$y\, d\sigma = C\, \left(\frac{s}{\sigma}\right)^{n-1} e^{-ns^2/2\sigma^2}\, d\sigma$$

from $\sigma = 0$ to $\sigma = \infty$ is $\dfrac{n^{\frac{1}{2}(n-2)}}{\Gamma[\frac{1}{2}(n-2)]\, 2^{\frac{1}{2}(n-4)}\, s}$

44. Show that the value of C required to normalize the curve

$$y\, d\sigma = C\, \left(\frac{a}{\sigma}\right)^{c+3} e^{-a^2/2\sigma^2}\, d\sigma$$

from $\sigma = 0$ to $\sigma = \infty$ is $\dfrac{1}{2^{\frac{1}{2}c}\, a\,\Gamma(\frac{1}{2}c + 1)}$. (This curve is Molina and Wilkinson's suggestion for a "prior existence" curve of σ; Bell System Technical Journal, vol. 8, 1929: pp. 632-645.)

45. The mean of the curve in the preceding exercise is at $E\sigma = [a\sqrt{c}/\sqrt{(2\pi)}]\ B(\frac{1}{2}\,\overline{c+1}, \frac{1}{2})$, and its second moment coefficient about the origin is $E\sigma^2 = a^2/c$.

46. The curve

$$y = y_o(1 + \alpha t)^{\frac{1-\alpha^2}{\alpha^2}}\ e^{-t/\alpha}$$

$$y_o = \alpha/(\alpha^2 e)^{\alpha^{-2}}\ \Gamma(\alpha^{-2})$$

is sometimes used to approximate the point binomial. This is equivalent to a Pearson Type III curve. A table of integrals for this curve was published by Salvosa.[13]

(a) Verify that the value given for y_o actually makes it a normalizing factor.

(b) Show that the mean of this curve is at $t = 0$; in other words, the origin is taken at the mean.

(c) The second moment coefficient of the curve about the origin is unity; hence the S.D. is unity.

(d) The third moment coefficient of the curve is 2α.

(e) The mode (maximum) is at $-\alpha$.

(f) If skewness be defined as $\dfrac{\text{mean-mode}}{\text{S. D.}}$, then the skewness of this curve is α.

(g) If skewness be defined as $\mu_3/2(\text{S.D.})^3$, then the skewness of this curve is α. μ_3 denotes the third moment coefficient of the curve about its mean. The two measures of skewness just given happen to coincide for the Type III curve, but for no other curve with skewness different from zero.